# EE 508
# Lecture 21

## Sensitivity Functions

- Comparison of Filter Structures
- Performance Prediction

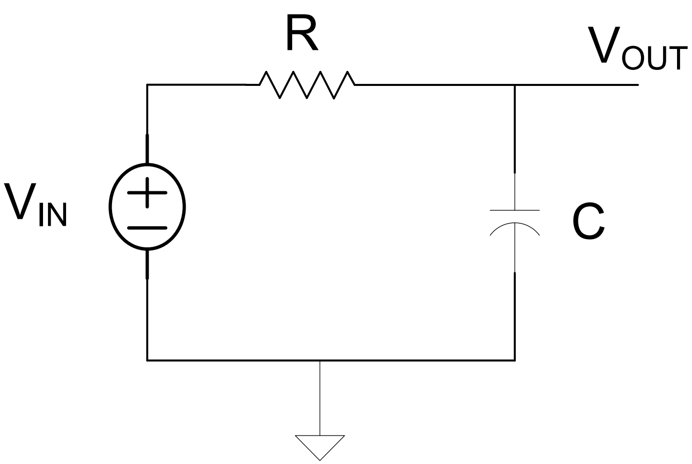
$$\frac{dF}{F} = \sum_{i=1}^{k}\left( \boxed{S^{f}_{x_i}\Big|_{\vec{X}_N}} \bullet \boxed{\frac{dx_i}{x_{iN}}} \right)$$

**Dependent on circuit structure**
(for some circuits, also not dependent on components)

**Dependent only on components**
(not circuit structure)

Review from last time

Consider:




$$T(s) = \frac{1}{1+RCs}$$

$$T(s) = \frac{\omega_0}{s+\omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

Review from last time

# Metrics for Comparing Circuits

## Summed Sensitivity

$$\rho_S = \sum_{i=1}^{m} S_{x_i}^{f}$$

**Not very useful because sum can be small even when individual sensitivities are large**

## Schoeffler Sensitivity

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

Strictly heuristic but does differentiate circuits with low sensitivities from those with high sensitivities

Review from last time

## Metrics for Comparing Circuits

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

**Often will consider several distinct sensitivity functions to consider effects of different components**

$$\rho_R = \sum_{All\ resistors} \left| S_{R_i}^{f} \right|$$

$$\rho_C = \sum_{All\ capacitors} \left| S_{C_i}^{f} \right|$$

$$\rho_{OA} = \sum_{All\ op\ amps} \left| \mathcal{S}_{\tau_i}^{f} \right|$$

Review from last time

Homogeniety (defn)

A function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ if

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Note: f may be comprised of more than n variables

## Review from last time

Theorem: If a function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ then

$$\sum_{i=1}^{n} S_{x_i}^{f} = m$$

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

The concept of homogeneity and this theorem were somewhat late to appear

Are there really any useful applications of this rather odd observation?

Let T(s) be a voltage or current transfer function

(i.e. dimensionless)

Observation: Impedance scaling does not change any of the following, provided Op Amps are ideal:

$T(s)$, $T(j\omega)$, $|T(j\omega)|$, $\omega_0$, $Q$, $p_k$, $z_k$

So, consider impedance scaling by a parameter λ

$$R \rightarrow \lambda R$$
$$L \rightarrow \lambda L$$
$$C \rightarrow C/\lambda$$

For these <u>impedance</u> <u>invariant</u> functions

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^0 f(x_1, x_2, \ldots x_n)$$

Thus, all of these functions are homogeneous of order m=0 in the impedances

Let T(s) be a Transresistance or Transconductance Transfer Function

Observation: Impedance scaling does not change any of the following, provided Op Amps are ideal:

$\omega_0$, Q, $p_k$, $z_k$, band edge

(these are impedance invariant functions)

So, consider impedance scaling by a parameter λ

$$R \rightarrow \lambda R$$
$$L \rightarrow \lambda L$$
$$C \rightarrow C/\lambda$$

For these <u>impedance</u> <u>invariant</u> functions

$$f\left(\lambda x_1, \lambda x_2, \ldots \lambda x_n\right) = \lambda^0 f\left(x_1, x_2, \ldots x_n\right)$$

Thus, all of these functions are homogeneous of order m=0 in the impedances

Theorem 1: If all op amps in a filter are ideal, then $\omega_o$, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Theorem 2: If all op amps in a filter are ideal and if T(s) is a dimensionless transfer function, T(s), T(jω), |T(jω)|, $\angle T(j\omega)$ , are homogeneous of order 0 in the impedances

Theorem 1: If all op amps in a filter are ideal, then $\omega_o$, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Proof of Theorem 1

These functions are all impedance invariant so if follows trivially that they are homogeneous of order 0 in all of the impedances

Theorem 3: If all op amps in a filter are ideal and if $T(s)$ is an impedance transfer function, $T(s)$ and $T(j\omega)$ are homogeneous of order 1 in the impedances

Theorem 4: If all op amps in a filter are ideal and if $T(s)$ is a conductance transfer function, $T(s)$ and $T(j\omega)$ are homogeneous of order -1 in the impedances

Corollary 1: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Corollary 2: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors then

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

Corollary 1: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Corollary 2: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors then for any pole or zero Q factor

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0 \qquad \sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

**Proof of Corollary 1:**

Corollary 1: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors and if a function f is homogeneous of order 0 in the impedances, then

$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Proof:

Since f is homogenous of order zero in the impedances, $z_1$, $z_2$, … $z_{k1+k2}$,

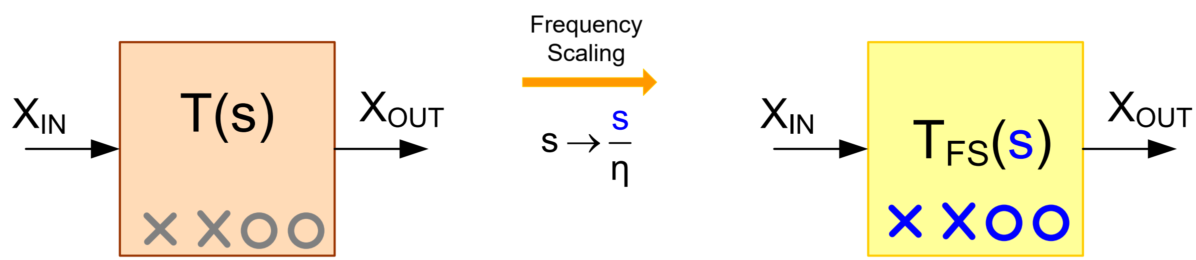
$$\sum_{i=1}^{k_1+k_2} S_{z_i}^{f} = 0$$

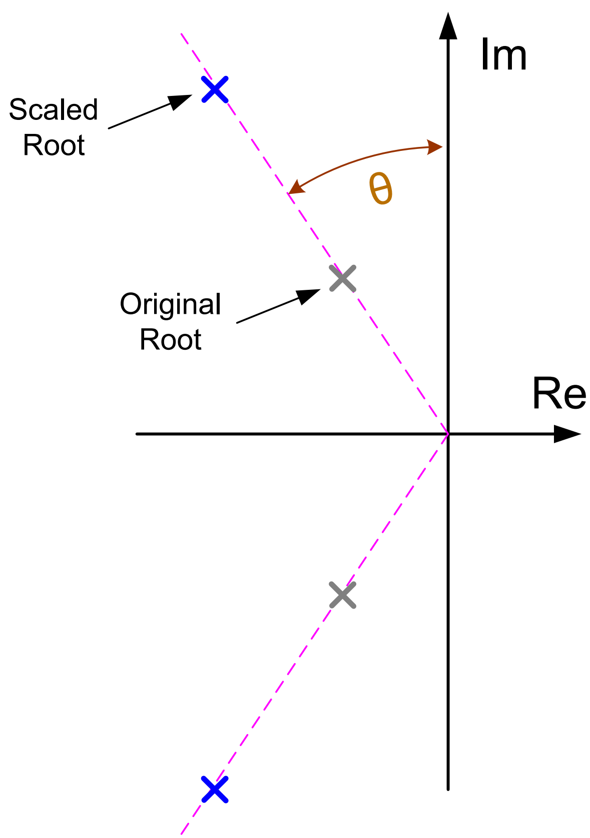
$$\therefore \quad \sum_{i=1}^{k_1} S_{R_i}^{f} + \sum_{i=1}^{k_2} S_{1/C_i}^{f} = 0$$

$$\therefore \quad \sum_{i=1}^{k_1} S_{R_i}^{f} - \sum_{i=1}^{k_2} S_{C_i}^{f} = 0$$

## Proof of Corollary 2:

Recall:




Frequency Scaling: Scaling all frequency-dependent elements by a constant

$$L \rightarrow \eta L$$

$$C \rightarrow \eta C$$

**Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus**

## Proof of Theorem:

$$T_{FS}(s) = T(s)\Big|_{s=\frac{s}{\eta}}$$

## Proof of Corollary 2:

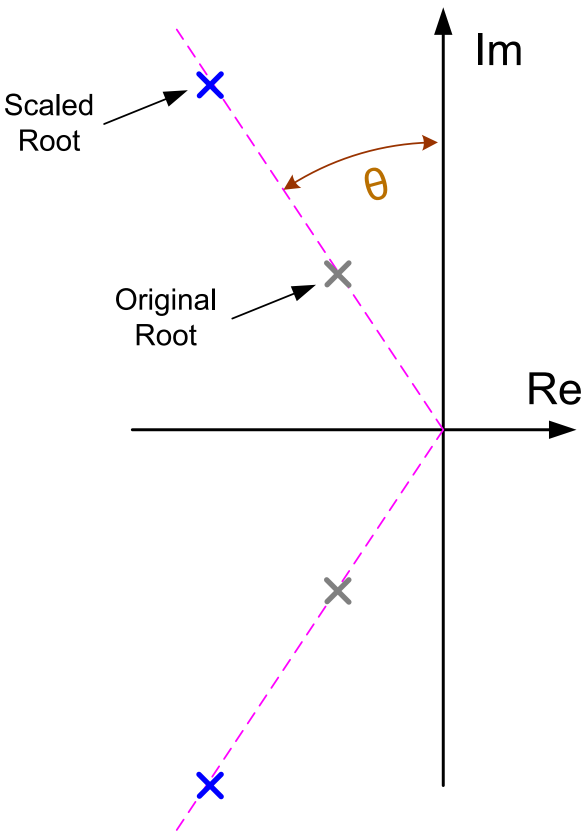
Recall:

**Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus**

**Proof:**

$$T_{FS}(s) = T(s)\Big|_{s=\frac{s}{\eta}}$$

Let p be a pole (or zero) of T(s)




$$T(p)=0 \qquad \text{consider} \qquad p = \frac{p}{\eta}$$

$$T_{FS}(s) = T\left(\frac{s}{\eta}\right) = T(s)$$

Since true for any variable, substitute in p

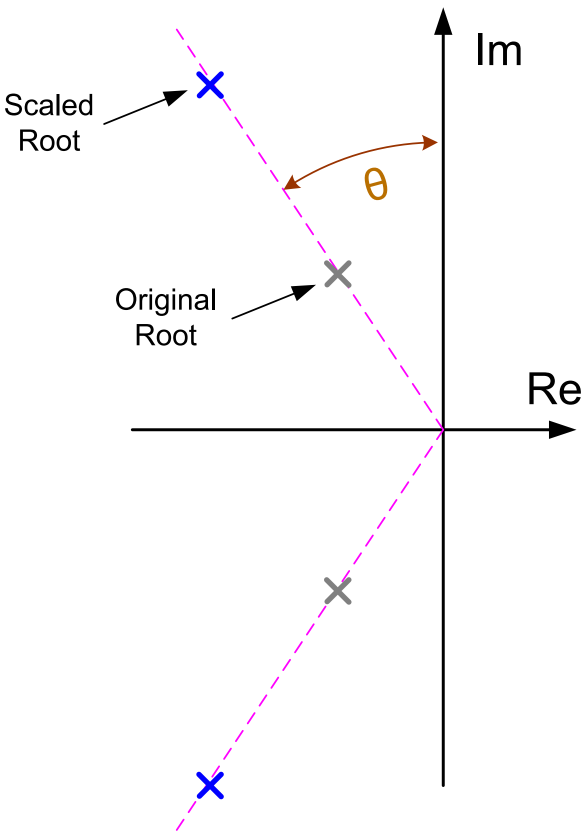
$$T_{FS}(p) = T\left(\frac{p}{\eta}\right) = T(p) = 0$$

Thus **p** is a pole (or zero) of $T_{FS}(s)$

## Proof of Corollary 2:

Recall:

**Theorem: If all components are frequency scaled, roots (poles and zeros) will move along a constant Q locus**




**Proof:** Thus **p** is a pole (or zero) of $T_{FS}(s)$

$$p = \frac{p}{\eta}$$

$$p = p\eta$$

Express **p** in polar form

$$p = re^{j\beta}$$

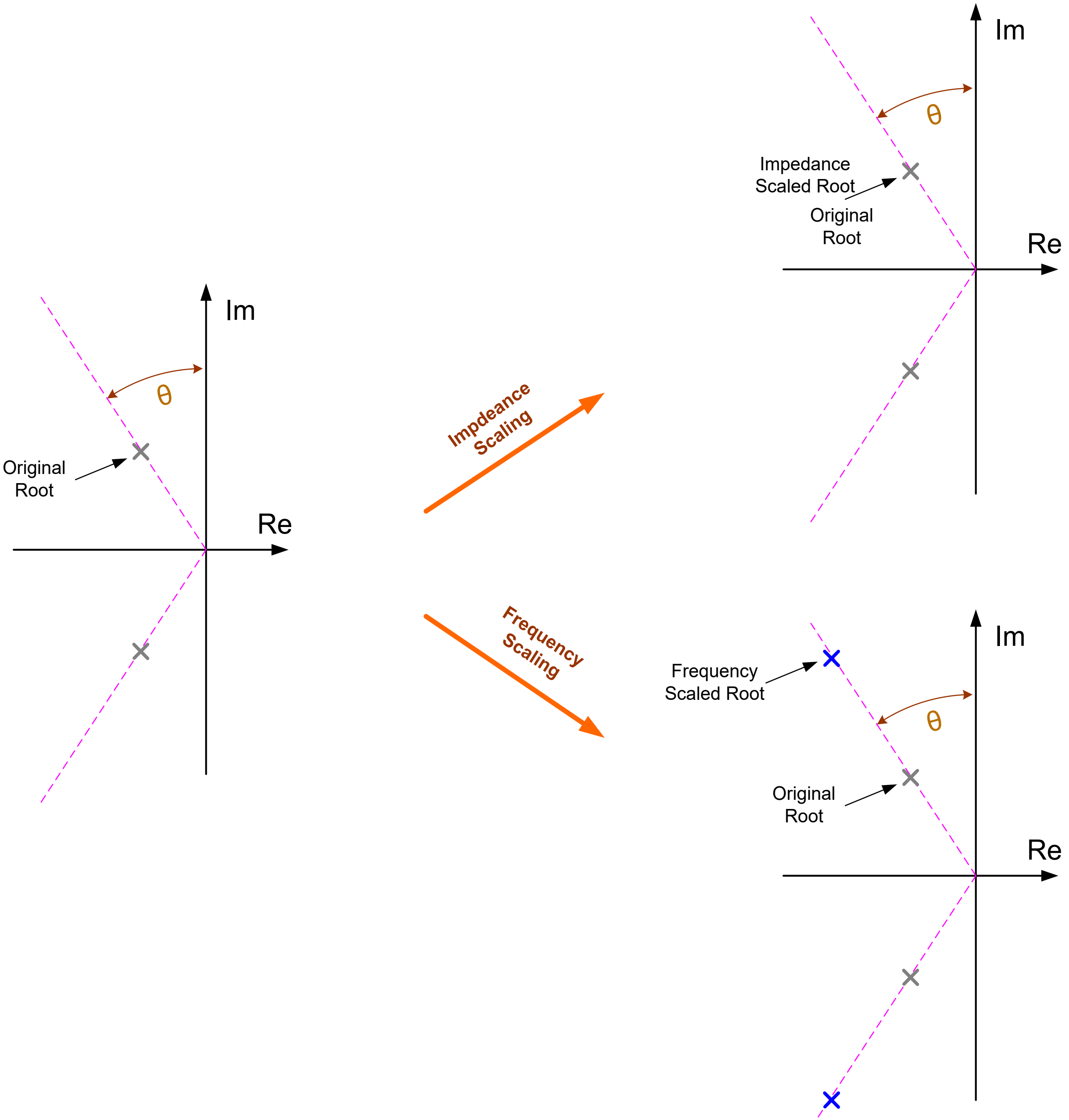
$$p = \eta p = \eta re^{j\beta}$$

Thus **p** and **p** have the same angle

Thus the scaled root has the same root Q

# Proof of Corollary 2: Impedance and Frequency Scaling

Recall:



## Proof of Corollary 2:

Corollary 2: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors then $\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$ and $\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$

Since impedance scaling does not change pole (or zero) Q, the pole (or zero) Q must be homogeneous of order 0 in the impedances

(For more generality, assume $k_3$ inductors)

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} + \sum_{i=1}^{k_2} S_{1/C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (1)$$

Since frequency scaling does not change pole (or zero) Q, the pole (or zero) Q must be homogeneous of order 0 in the frequency scaling elements

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (2)$$

## Proof of Corollary 2:

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} + \sum_{i=1}^{k_2} S_{1/C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (1)$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (2)$$

From theorem about sensitivity of reciprocals, can write (1) as

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} - \sum_{i=1}^{k_2} S_{C_i}^{Q} + \sum_{i=1}^{k_3} S_{L_i}^{Q} = 0 \qquad (3)$$

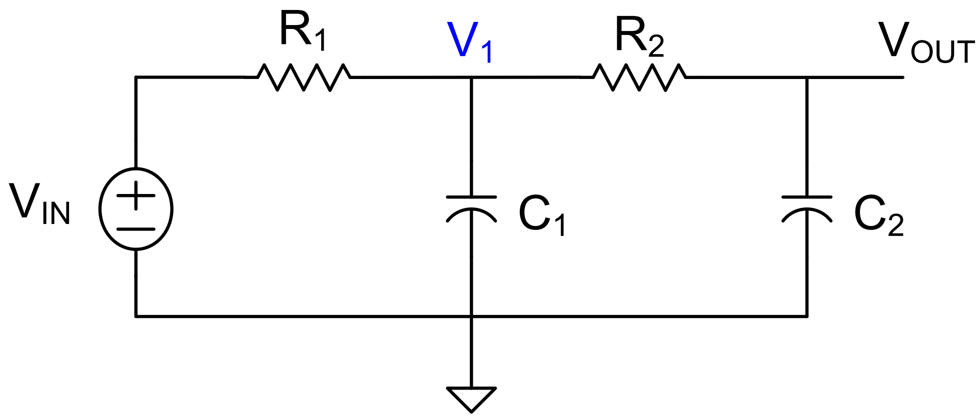
It follows from (2) and (3) that

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0 \qquad (4)$$

Since RC network, it follows from (4) and (2) that

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0 \qquad \sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

# Example




Determine the passive Q sensitivities

$$S_{R_1}^{Q} \quad S_{R_2}^{Q} \quad S_{C_1}^{Q} \quad S_{C_2}^{Q}$$

$$\left.\begin{aligned} V_{OUT}(sC_1+G_2) &= V_1G_2 \\ V_1(sC_1+G_1+G_2) &= V_{IN}G_1 + V_{OUT}G_2 \end{aligned}\right\}$$

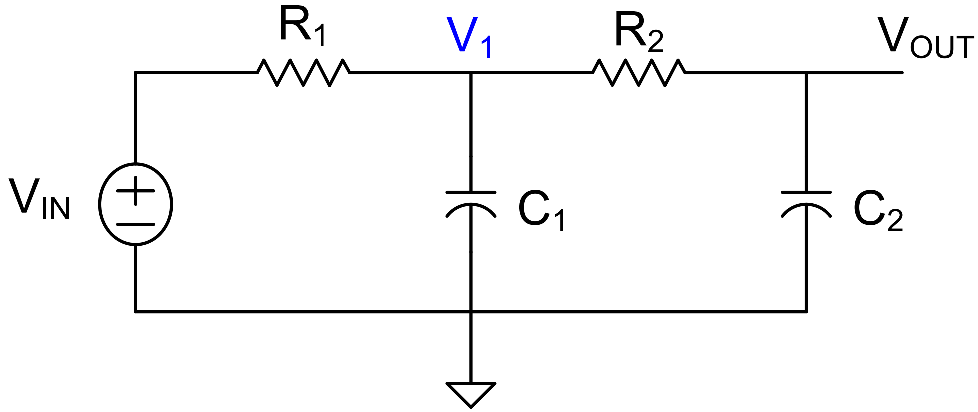
$$T(s) = \frac{1}{s^2(R_1R_2C_1C_2)+s(R_1C_1+R_1C_2+R_2C_2)+1}$$

$$\omega_0 = \frac{1}{\sqrt{R_1R_2C_1C_2}} \qquad Q = \frac{\sqrt{R_1R_2C_1C_2}}{R_1C_1+R_1C_2+R_2C_2}$$

By the definition of sensitivity, it follows that

$$S_{R_1}^{Q} = \frac{(R_1C_1+R_1C_2+R_2C_1)\frac{1}{2}(R_1R_2C_1C_2)^{-1/2}R_2C_1C_2-(C_1+C_2)(R_1R_2C_1C_2)^{1/2}}{(R_1C_1+R_1C_2+R_2C_2)^2}\bullet\frac{R_1}{Q}$$

# Example



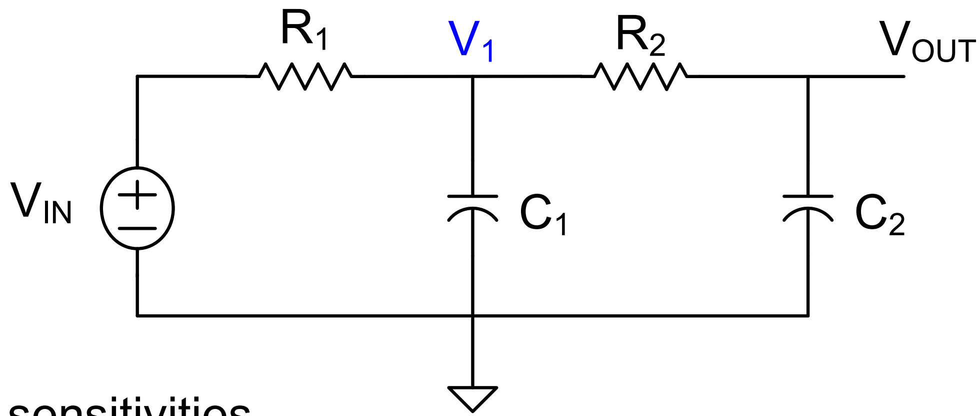

Determine the passive Q sensitivities $S_{R_1}^{Q}$ $S_{R_2}^{Q}$ $S_{C_1}^{Q}$ $S_{C_2}^{Q}$

$$S_{R_1}^{Q}=\frac{\left(R_1C_1+R_1C_2+R_2C_1\right)\frac{1}{2}\left(R_1R_2C_1C_2\right)^{-1/2}R_2C_1C_2-\left(C_1+C_2\right)\left(R_1R_2C_1C_2\right)^{1/2}}{\left(R_1C_1+R_1C_2+R_2C_2\right)^{2}}\bullet\frac{R_1}{Q}$$

Following some tedious manipulations, this simplifies to

$$S_{R_1}^{Q}=\frac{1}{2}-\frac{R_1\left(C_1+C_2\right)}{R_1C_1+R_1C_2+R_2C_2}$$

# Example




Determine the passive Q sensitivities

Following the same type of calculations, can obtain

$$S^{Q}_{R_1}=\frac{1}{2}-\frac{R_1\left(C_1+C_2\right)}{R_1C_1+R_1C_2+R_2C_2} \qquad S^{Q}_{R_2}=\frac{1}{2}-\frac{R_2C_2}{R_1C_1+R_1C_2+R_2C_2}$$

$$S^{Q}_{C_1}=\frac{1}{2}-\frac{R_1C_1}{R_1C_1+R_1C_2+R_2C_2} \qquad S^{Q}_{C_2}=\frac{1}{2}-\frac{C_2\left(R_1+R_2\right)}{R_1C_1+R_1C_2+R_2C_2}$$

Verify

$$\sum_{i=1}^{k_2} S^{Q}_{C_i}=0 \qquad \sum_{i=1}^{k_1} S^{Q}_{R_i}=0$$

Could have saved considerable effort in calculations by using these theorems after $S^{Q}_{R_1}$ and $S^{Q}_{C_1}$ were calculated

Corollary 3: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors and if $p_k$ is any pole and $z_h$ is any zero, then

$$\sum_{i=1}^{k_1} S_{R_i}^{p_k} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{p_k} = -1$$

and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

Corollary 3: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors and if $p_k$ is any pole and $z_h$ is any zero, then

$$\sum_{i=1}^{k_1} S_{R_i}^{p_k} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{p_k} = -1$$

and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

**Proof:**

It was shown that scaling the frequency dependent elements by a factor η divides the pole (or zero) by η

Thus roots (poles and zeros) are homogeneous of order -1 in the frequency scaling elements

**Proof:**

Thus roots (poles and zeros) are homogeneous of order -1 in the frequency scaling elements

(For more generality, assume $k_3$ inductors)

$$\sum_{i=1}^{k_2} S_{C_i}^{p} + \sum_{i=1}^{k_3} S_{L_i}^{p} = -1 \qquad (1)$$

Since impedance scaling does not affects the poles, they are homogenous of order 0 in the impedances

$$\sum_{i=1}^{k_1} S_{R_i}^{p} + \sum_{i=1}^{k_2} S_{1/C_i}^{p} + \sum_{i=1}^{k_3} S_{L_i}^{p} = 0 \qquad (2)$$

Since there are no inductors in an active RC network, is follows from (1) that

$$\sum_{i=1}^{k_2} S_{C_i}^{p} = -1$$

And then from (2) and the theorem about sensitivity to reciprocals that

$$\sum_{i=1}^{k_1} S_{R_i}^{p} = -1$$

Corollary 4: If all op amps in an RC active filter are ideal and there are $k_1$ resistors and $k_2$ capacitors and if $Z_{IN}$ is any input impedance of the network, then

$$\sum_{i=1}^{k_1} S_{R_i}^{Z_{IN}} - \sum_{i=1}^{k_2} S_{C_i}^{Z_{IN}} = 1$$

Claim: If op amps in the filters considered previously are not ideal but are modeled by a gain $A(s)=1/(\tau s)$, then all previous summed sensitivities developed for ideal op amps hold provided they are evaluated at the nominal value of $\tau=0$

# Sensitivity Analysis

If a closed-form expression for a function f is obtained, a straightforward but tedious analysis can be used to obtain the sensitivity of the function to any components

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

---

Consider:

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

Closed-form expressions for $T(s)$, $T(j\omega)$, $|T(j\omega)|$, $\angle T(j\omega)$, $a_i$, $b_i$, can be readily obtained

# Sensitivity Analysis

If a closed-form expression for a function f is obtained, a straightforward but tedious analysis can be used to obtain the sensitivity of the function to any components

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

---

Consider:

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

Closed-form expressions for $p_i$, $z_i$, pole or zero Q, pole or zero $\omega_0$, peak gain, $\omega_{3dB}$, BW, … (generally the most critical and useful circuit characteristics) are difficult or impossible to obtain !

# Bilinear Property of Electrical Networks

Theorem: Let x be any component or Op Amp time constant (1st order Op Amp model) of any linear active network employing a finite number of amplifiers and lumped passive components. Any transfer function of the network can be expressed in the form

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

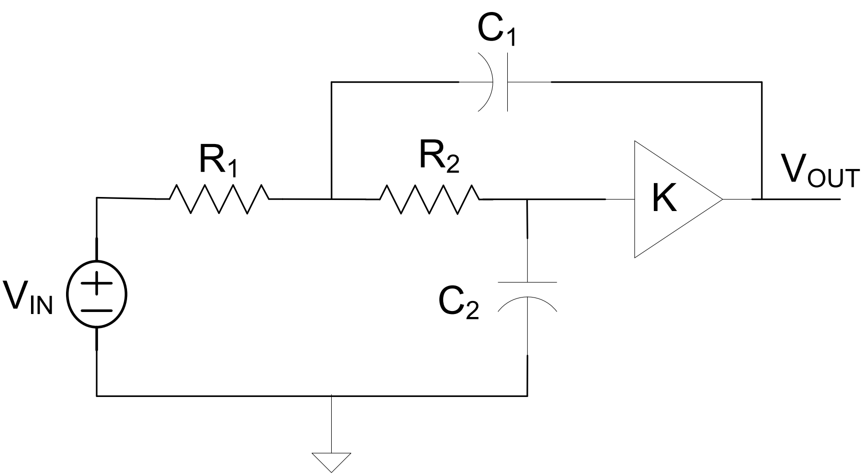
where $N_0$, $N_1$, $D_0$, and $D_1$ are polynomials in s that are not dependent upon x

A function that can be expressed as given above is said to be a bilinear function in the variable x and this is termed a bilateral property of electrical networks.

The bilinear relationship is useful for

1. Checking for possible errors in an analysis
2. Pole sensitivity analysis

# Example of Bilinear Property :    +KRC Lowpass Filter



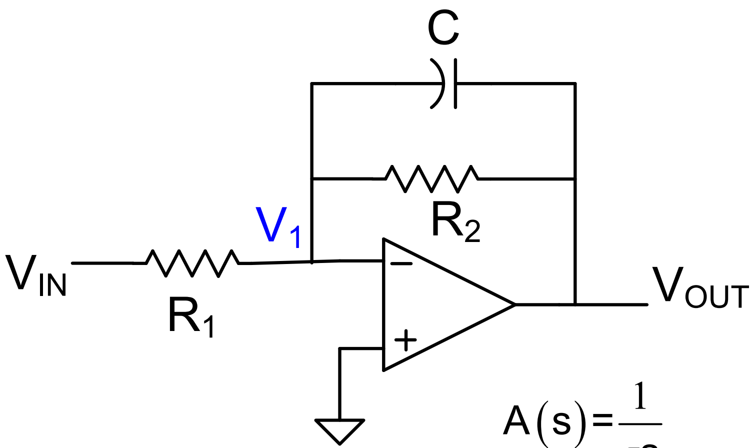

$$T(s)=\frac{\dfrac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{(1-K_0)}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}+K_0\tau s\left(s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}\right)}$$

## Consider $R_1$

$$T(s)=\frac{\dfrac{K_0}{R_2C_1C_2}}{R_1s^2+s\left[\dfrac{1}{C_1}+R_1\dfrac{1}{R_2C_1}+R_1\dfrac{(1-K_0)}{R_2C_2}\right]+\dfrac{1}{R_2C_1C_2}+K_0\tau s\left(R_1s^2+s\left[\dfrac{1}{C_1}+R_1\dfrac{1}{R_2C_1}+R_1\dfrac{1}{R_2C_2}\right]+\dfrac{1}{R_2C_1C_2}\right)}$$

$$T(s)=\frac{\left[\dfrac{K_0}{R_2C_1C_2}\right]+R_1\bullet[0]}{\left[s\dfrac{1}{C_1}+\dfrac{1}{R_2C_1C_2}+K_0\tau s\left(s\dfrac{1}{C_1}\right)+\dfrac{1}{R_2C_1C_2}\right]+R_1\left[s^2+s\left[\dfrac{1}{R_2C_1}+\dfrac{(1-K_0)}{R_2C_2}\right]+K_0\tau s\left(s^2+s\left[\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right]\right)\right]}$$

# Example of Bilinear Property




$$A(s)=\frac{1}{\tau s}$$

$$\left.\begin{aligned} &V_1(G_1+G_2+sC) = V_{IN}G_1+V_{OUT}(sC+G_2) \\ &V_{OUT} = -V_1\left(\frac{1}{\tau s}\right) \end{aligned}\right\}$$

$$T(s) = \frac{-R_2}{R_1+R_1R_2Cs+\tau s(sCR_1R_2+R_1+R_2)}$$

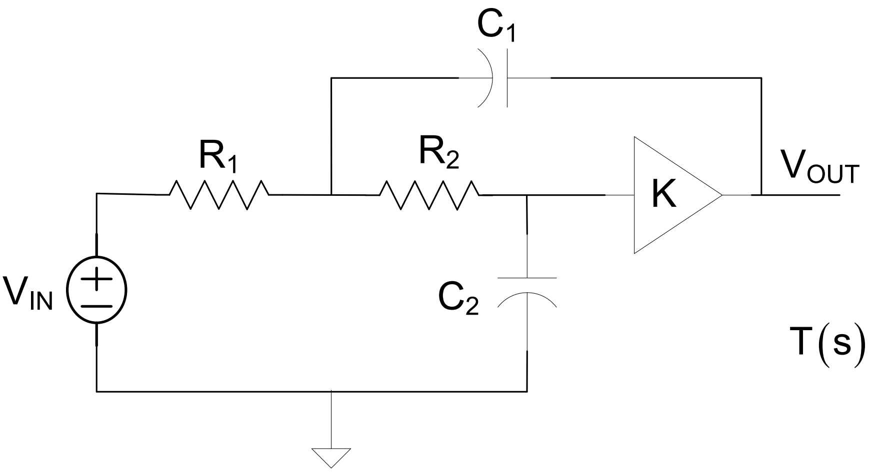
Consider $R_1$

$$T(s) = \frac{-R_2+0\bullet R_1}{[\tau sR_2]+R_1[1+R_2Cs+\tau s(sCR_2+1)]}$$

Consider τ

$$T(s) = \frac{-R_2+0\bullet\tau}{[R_1(1+R_2Cs)]+\tau[sR_2+sR_1(sCR_2+1)]}$$

# Example of Bilinear Property : +KRC Lowpass Filter




Equal R Equal C

$$T(s)=\frac{\dfrac{K_0}{R^2C^2}}{s^2+s\left[\dfrac{(3\text{-}K_0)}{RC}\right]+\dfrac{1}{R^2C^2}+K_0\tau s\left(s^2+s\left[\dfrac{3}{RC}\right]+\dfrac{1}{R^2C^2}\right)}$$

$$T(s)=\frac{K_0}{R^2\left(C^2s^2+K_0\tau sC^2\right)+R\left(sC(3\text{-}K_0)+3K_0C\tau s^2\right)+(1+K_0\tau s)}$$

Can not eliminate the $R^2$ term

- Bilinear property only applies to individual components
- Bilinear property was established only for T(s)

Stay Safe and Stay Healthy !

End of Lecture 21